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ON CERTAIN DIOPHANTINE EQUATIONS HAVING MULTIPLE PARAMETER SOLUTIONS.[1]

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1. The object of this note is to obtain a multiple parameter solution of each of several Diophantine equations, as follows:

$$\begin{array}{rr} x^3 + y^3 + z^3 = 2t^3, & x^3 + y^3 + z^3 = 2t^{9k}, \\ x^3 + y^3 + z^3 + u^3 = 3t^3, & x^3 + y^3 + z^3 + u^3 = kt^{\alpha}, \\ x^3 + 2y^3 + 3z^3 = t^3, & x^3 + 2y^{3\alpha} + 3z^{3\beta} = t^3, \\ x^4 + y^4 + 4z^4 = t^4, & x^4 + 2y^4 + 2z^4 = t^4, \\ x^4 + 8y^4 + 8z^4 = t^4, & x^4 + y^4 + z^4 = 2t^4, \end{array}$$

where k is a positive integer and α and β are odd positive integers. The methods employed are altogether elementary in their character.

2. If in the equation[2]

$$x^3 + y^3 + z^3 = 2t^3, \tag{1}$$

the numbers x, y, z, t have the common divisor d, then the equation may be divided through by d^3 and the resulting equation will be of the same form as (1). Hence it is sufficient to seek only those solutions of (1) in which x, y, z, t have the greatest common divisor unity; and consequently we confine attention to this case. It is obvious, then, that two of the numbers x, y, z, are odd while the third one is even. If we suppose x and y to be odd, we may write

$$x = r + s, \quad y = r - s,$$

where r and s are integers.

Substituting these values of x and y in (1) we have

$$2r^3 + 6rs^2 = 2t^3 - z^3.$$

This equation will be satisfied if we have simultaneously $r = t$, $z = -6rs^2$. If we write r in the form $r = k\rho^3$ where k has no cube factor except unity, then we must have $s = \pm k\sigma^3$. It is obvious that in the resulting solution the numbers x, y, z, t have the common factor k; and hence, for our present case, we have $k = 1$. Our desired solution of (1) then is

$$x = \rho^3 \pm 6\sigma^3,\ y = \rho^3 \pm 6\sigma^3,\ z = -6\rho\sigma^2,\ t = \rho^3. \tag{2}$$

It should be noticed that there is nothing in the argument to indicate whether (2) affords the general solution of (1). A similar remark applies to the other problems treated in this note.

It may be observed in passing that we have incidentally obtained a two-parameter solution of the equation

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[1] Presented to the American Mathematical Society, March 21, 1913.

[2] A single solution of this equation is given in the French Encyclopédie, tome I, vol. 3, p. 210; namely, $x = 7$, $y = 11$, $z = 102$, $t = 81$. This is not a special case of the solution obtained in the text.

$$x^3 + y^3 + z^3 = 2t^9,$$

x, y, z having the same values as before and t being equal to ρ. By taking ρ equal to the kth power of an integer we have further a two-parameter solution of the equation

$$x^3 + y^3 + z^3 = 2t^{9k}.$$

3. We shall now obtain solutions of the equation

$$x^3 + y^3 + z^3 + u^3 = 3t^3. \tag{3}$$

If we write

$$x = a + b - c, \quad y = a - b + c, \quad z = -a + b + c,$$

we have

$$a = \tfrac{1}{2}(x + y), \quad b = \tfrac{1}{2}(x + z), \quad c = \tfrac{1}{2}(y + z),$$

so that a number of the set a, b, c either is an integer or is the half of an odd integer. Substituting these values of x, y, z in (3) we obtain

$$(a + b + c)^3 - 24\,abc + u^3 = 3t^3.$$

This equation will be satisfied if we have simultaneously $u = -(a + b + c)$, $t^3 = -8abc$. If we write $a = m\alpha^3$, $b = n\beta^3$, $mn = ks^3$, where neither m nor n nor k has a cubic factor greater than 1, we have $c = \pm k^2\gamma^3$. Therefore we have the following five-parameter solution of (3)

$$x = m\alpha^3 + n\beta^3 \pm k^2\gamma^3, \quad y = m\alpha^3 - n\beta^3 \pm k^2\gamma^3, \quad z = -m\alpha^3 + n\beta^3 \pm k^2\gamma^3, \tag{4}$$

$$u = -(m\alpha^3 + n\beta^3 \pm k^2\gamma^3), \quad t = \pm 2ks\alpha\beta\gamma,$$

where $mn = ks^3$. The parameters entering into the solution represent either integers or the halves of odd integers, the latter being so related that the values of x, y, z, u, t are all integers.

It is clear that we may also readily obtain a five-parameter solution of the equation

$$x^3 + y^3 + z^3 + u^3 = kt^\alpha,$$

where α is any positive odd integer and k is any integer.

4. In the equation

$$x^3 + 2y^3 + 3z^3 = t^3, \tag{5}$$

let us make the substitution

$$x = r - s, \quad t = r + s;$$

then we have $2y^3 + 3z^3 = 2s^3 + 6r^2s$. This will be satisfied if we have simultaneously $y = s$, $z^3 = 2r^2s$. We exclude the case when r and s have a common factor greater than unity, since this would lead to a solution of x, y, z, t with such a common factor. Then it is clear that we have $s = m^3$, $r = 2n^3$. Hence we obtain the two-parameter solution

$$x = 2n^3 - m^3, \quad y = m^3, \quad z = 2mn^2, \quad t = 2n^3 + m^3. \tag{6}$$

It is easy to see that we have obtained incidentally a two-parameter solution of the equation

$$x^3 + 2y^{3\alpha} + 3z^{3\beta} = t^3, \tag{7}$$

where α and β are any odd positive integers. For, let μ be the least common multiple of α and β and write

$$m = r^{\mu}, \quad 2n^2 = s^{\mu} = (2\cdot\sigma^2)^{\mu},$$

whence $n = 2^{\frac{\mu-1}{2}}\sigma^{\mu}$. These values of m and n, taken in connection with (6), lead to the following solution of (7):

$$x = 2^{\frac{3\mu-1}{2}}\sigma^{3\mu} - r^{3\mu}, \quad y = r^{\frac{3\mu}{\alpha}}, \quad z = 2^{\frac{\mu}{\beta}}\sigma^{\frac{2\mu}{\beta}}r^{\frac{\mu}{\beta}}, \quad t = 2^{\frac{3\mu-1}{2}}\sigma^{3\mu} + r^{3\mu}. \tag{8}$$

As an analogue of equation (5) we have the equation

$$x^2 + 2y^2 = z^2$$

with the solution

$$x = 2t^2 - s^2, \quad y = 2st, \quad z = 2t^2 + s^2.$$

5. If for the equation

$$x^4 + y^4 + 4z^4 = t^4 \tag{9}$$

we confine attention to the case in which x, y, z, t have the greatest common divisor 1, it is easy to see that t is odd and either x or y is odd. We shall suppose that x is odd. Then if we write

$$x = r - s, \quad t = r + s,$$

it follows that r and s are integers. Substituting these values in (9) we have

$$y^4 + 4z^4 = 8r^3s + 8rs^3.$$

This equation will be satisfied if we have simultaneously $y^4 = 8r^3s$, $4z^4 = 8rs^3$ or $y^4 = 8rs^3$, $4z^4 = 8r^3s$. It is clear that these two cases will not lead to different solutions of (9).

Let us consider the first case: $y^4 = 8r^3s$, $4z^4 = 8rs^3$. If $r = k\rho^4$, where k has no fourth power factor except unity, then $s = 2k\sigma^4$, as we see from the first equation. Hence $4z^4 = 64k^4\rho^4\sigma^{12}$. If x, y, z, t are to have no common factor (except unity) it is clear that $k = 1$. Hence we are led to the following solution of (9):

$$x = \rho^4 - 2\sigma^4, \quad y = 2\rho^3\sigma, \quad z = 2\rho\sigma^3, \quad t = \rho^4 + 2\sigma^4. \tag{10}$$

6. Proceeding in a similar way, we obtain for the equations

$$x^4 + 2y^4 + 2z^4 = t^4,$$
$$x^4 + 8y^4 + 8z^4 = t^4,$$
$$x^4 + y^4 + 2z^4 = 2t^4,$$
$$x^4 + y^4 + 8z^4 = 8t^4,$$

the solutions

$$\begin{array}{llll} x = 4\rho^4 - \sigma^4, & y = 4\rho^3\sigma, & z = 2\rho\sigma^3, & t = 4\rho^4 + \sigma^4, \\ x = \rho^4 - \sigma^4, & y = \rho^3\sigma, & z = \rho\sigma^3, & t = \rho^4 + \sigma^4, \\ x = 2\rho^3\sigma, & y = 2\rho\sigma^3, & z = \rho^4 - \sigma^4, & t = \rho^4 + \sigma^4, \\ x = 8\rho^3\sigma, & y = 4\rho\sigma^3, & z = 4\rho^4 - \sigma^4, & t = 4\rho^4 + \sigma^4, \end{array}$$

respectively.

7. Let us consider next the equation

$$x^4 + y^4 + z^4 = 2t^4. \tag{11}$$

We shall seek solutions of this equation in which x, y, z, t are polynomials in an integral parameter k, the degree being not greater than 2. It is easy to see that the coefficients of k^2 in these polynomials must satisfy equation (11). The corresponding statement is true also of the independent terms. Now (11) has the obvious solutions obtained by putting one of the variables in the first member equal to zero and giving to the other variables in any way the values $+ 1$ or $- 1$. This gives us a certain number of cases to examine in order to see whether the coefficients of powers of k in the several polynomials can be so determined that (11) becomes an identity on substituting for the variables the polynomials in k.

Thus, as one case, we have to examine whether the ambiguous signs can be suitably chosen and the coefficients a, b, c, d so determined that

$$(k^2 + ak)^4 + (bk \pm 1)^4 + (k^2 + ck \pm 1)^4 = 2(k^2 + dk \pm 1)^4$$

is an algebraic identity in k. Carrying out the necessary reckoning in each of the possible cases which may arise we have the following solution of (11):

$$x = k^2 - 2k, \quad y = 2k - 1, \quad z = k^2 - 1, \quad t = k^2 - k + 1. \tag{12}$$

The reckoning is facilitated by observing that when k is replaced by $k + 1$ the resulting independent terms must still form a solution of (11).

THE CUBE ROOT OF A BINOMIAL SURD.

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In Wells's *Text Book in Algebra*, page 548, the equation $x^3 + x - 2 = 0$ leads by Cardan's method to the solution $\frac{1}{3}(\sqrt[3]{27 + 6\sqrt{21}} + \sqrt[3]{27 - 6\sqrt{21}})$, and the author implies that this cannot be simplified, for he says "there is no method in algebra for finding the cube root of a binomial surd."

The challenge suggested by this problem led to the following discussion, which, so far as the writer can learn, has not been published heretofore. Namely, to determine whether a quadratic binomial surd has a cube root which can be obtained by algebraic methods; and, when possible, to obtain this root.

Given the quadratic binomial surd $a + b\sqrt{c}$ in which a, b and c are positive